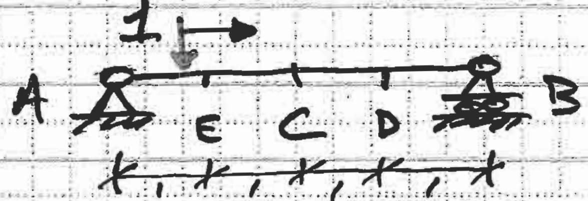
Influence Lines

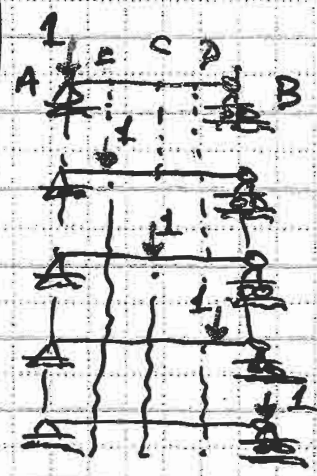
An influence line is a graph of the variation of a particular load effect, at a specific location in the structure, as a unit load traverses the structure.

Load effect $\rightarrow M, V, F, \delta \ldots$ etc.

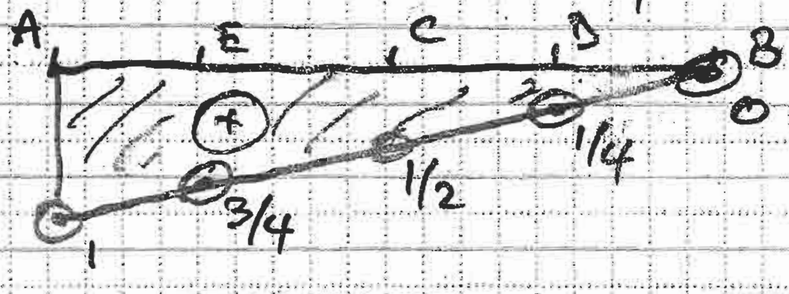
As an example, consider this beam:



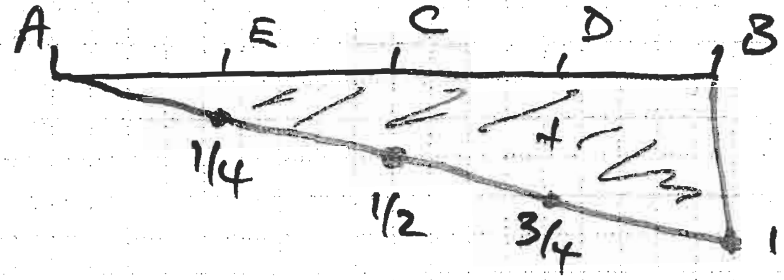
Unit load at:	V_A	V_B	M_C
A	1	0	0
E	3/4	1/4	1/2
C	1/2	1/2	1
D	1/4	3/4	1/2
B	0	1	0



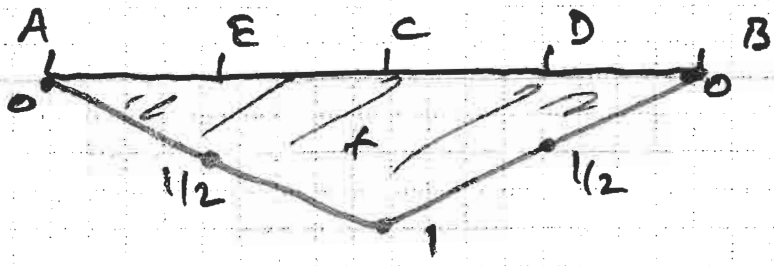
Thus we draw the I.L. for V_A as:



Similarly for V_B:



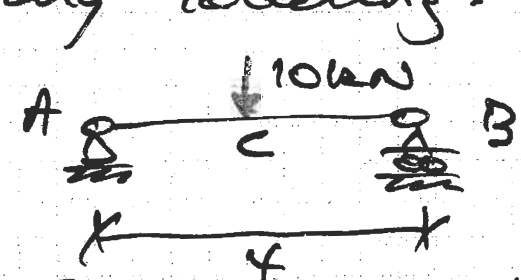
And the I.L. for M_C is:



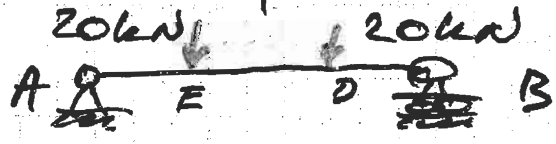
From the influence lines, by the principle of superposition, we can calculate the values of V_A, V_B & M_C when the beam AB is subjected to any arrangement of force-loads.

For example, calculate V_A, V_B & M_C for the following loading:

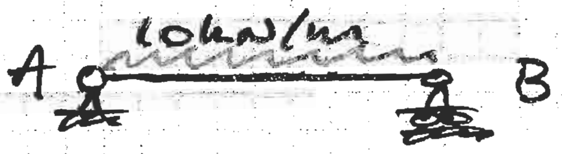
(1)



(2)



(3)



Consider V_A for case (1). We see, off the influence line for V_A, that when the unit load (1kN) was at the same position as the 10kN load (i.e. was located at C), it gave:

$$\text{For } 1\text{ kN @ C} \qquad V_A = 1/2$$

$$\Rightarrow \text{For } 10\text{ kN @ C} \qquad V_A = \tfrac{1}{2} \times 10$$
$$= 5\text{ kN}$$

This is as we would obviously expect.

Thus: to obtain the load effect we multiply the influence ordinate at the location of the load by the value of the load.

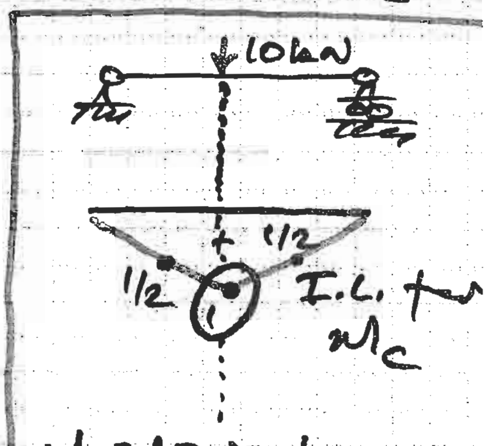
Thus, for (1):

$$V_B = \underbrace{1/2}_{\text{I.L. ordinate @ C}} \times \underbrace{10}_{\text{Value of load}} = 5\text{kN}$$

value of reaction at B due to 10kN load @C

and

$$M_C = 1 \times 10 = 10\text{kNm}$$



It can help to visualise the diagrams beside each other as shown → $M_C = 10 \times 1$

For Case (2) and for V_A we see that for the 20kN load at E, the I.L. ordinate is 3/4, thus:

$$V_{A\,(20kN @ E)} = \tfrac{3}{4} \times 20 = 15\text{ kN}$$

and for the 20kN at D, we have:

$$V_{A\,(20kN @ D)} = \tfrac{1}{4} \times 20 = 5\text{ kN}$$

Obviously the actual reaction is the sum of the effects of both loads:

$$V_A = 15 + 5 = 20\text{kN} \quad \text{(as expected)}$$

$$= \underbrace{\tfrac{3}{4} \times 20}_{E} + \underbrace{\tfrac{1}{4} \times 20}_{D} \quad \text{(by I.L. only)}$$

Thus for V_B we have:

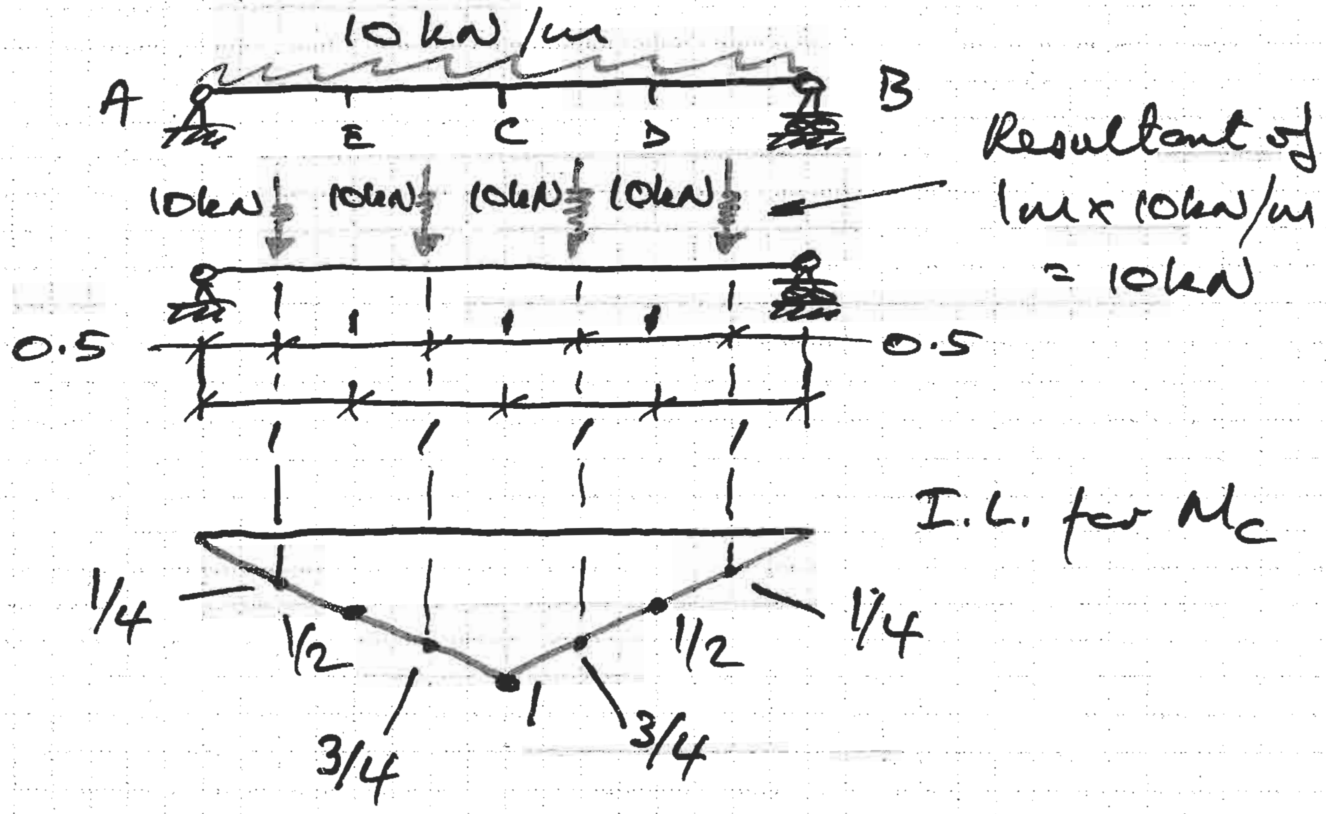
$$V_B = \underbrace{\tfrac{1}{4} \times 20}_{E} + \underbrace{\tfrac{3}{4} \times 20}_{D} = 20\text{ kN}$$

and for M_c:

$$M_c = \underbrace{\tfrac{1}{2} \times 20}_{E} + \underbrace{\tfrac{1}{2} \times 20}_{D} = 20\text{kNm}$$

Thus for multiple loads we sum the loads × I.L. ordinate for the total value of the load effect.

For Case (3) and M_C, examine the following:



From the result of case (2) we can see:

$$M_C = \overset{AE}{\overline{10 \times 1/4}} + \overset{EC}{\overline{10 \times \tfrac{3}{4}}} + \overset{CD}{\overline{10 \times \tfrac{3}{4}}} + \overset{DB}{\overline{10 \times 1/4}}$$

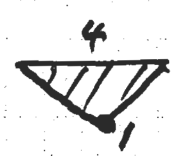
$$= 20 \text{ kNm}$$

Note that under each 10kN resultant we have used: 10kN × 1m × I.L. ordinate

= 10 × (Area of I.L. over 1m)

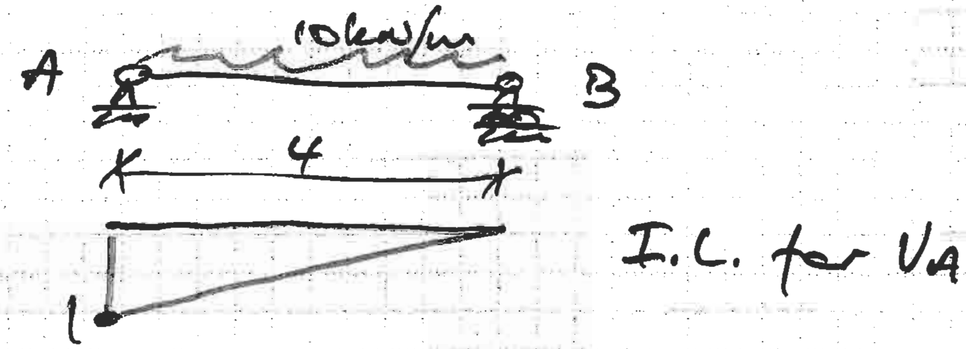
Also, as we have added all of the 10kN:

M_C = 10 × (Area of I.L. over full length)

In this case: Area of  I.L. for M_C

$$M_C = 10 \times \left[\tfrac{1}{2} \times 1 \times 4\right] = 20 \text{ kNm}$$

Continuing Case (3) for V_A we have:



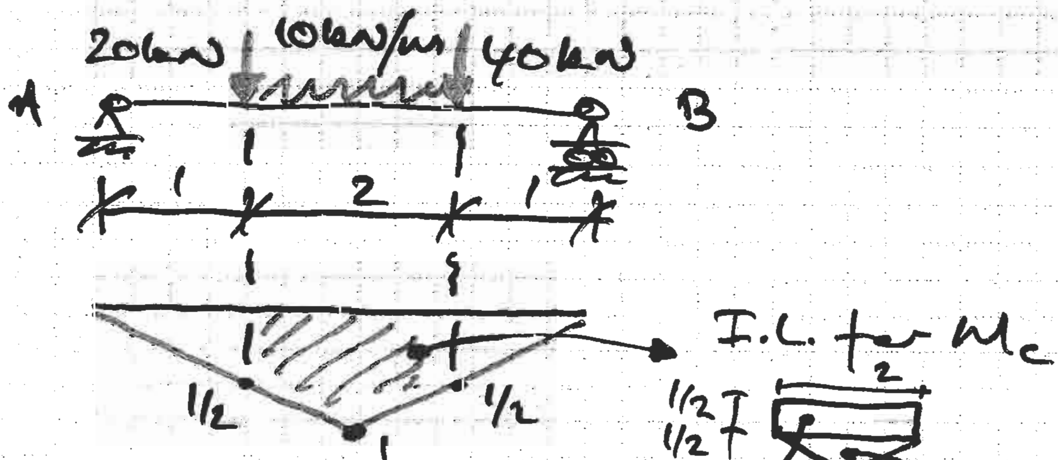
$$\Rightarrow V_A = 10 \times \left[\tfrac{1}{2} \times 4 \times 1\right] = 20\text{kN}$$

And also for V_B:

$$V_B = 10 \times \left[\tfrac{1}{2} \times 4 \times 1\right] = 20\text{kN}$$

Note that for partial UDL's we only take the area under the UDL:

(4)



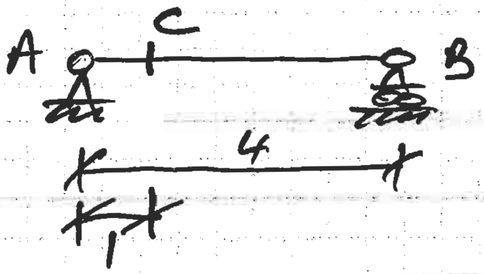
$$M_C = 20 \times \tfrac{1}{2} + 40 \times \tfrac{1}{2} + 10 \times \left[2 \times \tfrac{1}{2} + \tfrac{1}{2} \cdot 2 \cdot \tfrac{1}{2}\right]$$
$$= 45\text{kNm}$$

$$V_A = 20 \times \tfrac{3}{4} + 40 \times \tfrac{1}{4} + 10 \times \left[2 \times \tfrac{1}{4} + \tfrac{1}{2} \cdot 2 \cdot \tfrac{1}{2}\right]$$
$$= 35\text{kN}$$

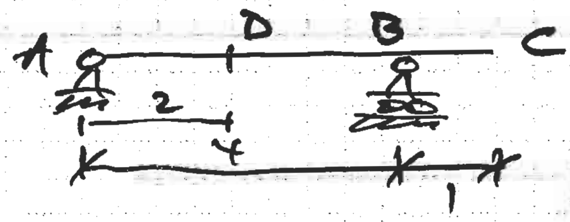
$$V_B = 20 \times \tfrac{1}{4} + 40 \times \tfrac{3}{4} + 10 \times \left[2 \times \tfrac{1}{4} + \tfrac{1}{2} \cdot 2 \cdot \tfrac{1}{2}\right]$$
$$= 45\text{kN}$$

It is clear to see that influence lines may be a much quicker way of calculating critical design values for a structure subjected to many different loading arrangements.

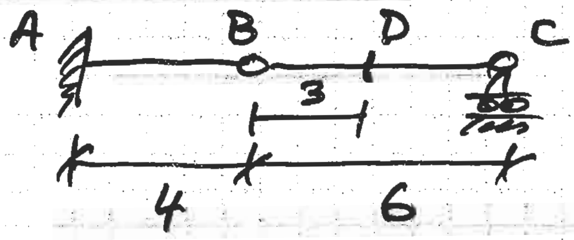
Problems

(1) 

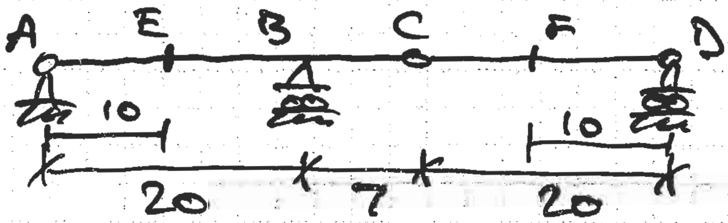
Draw the I.L. for V_C and M_C

(2) 

Draw the I.L. for V_B, M_B & M_D

(3) 

Draw the I.L. for M_A, V_A, V_B, M_D & V_C

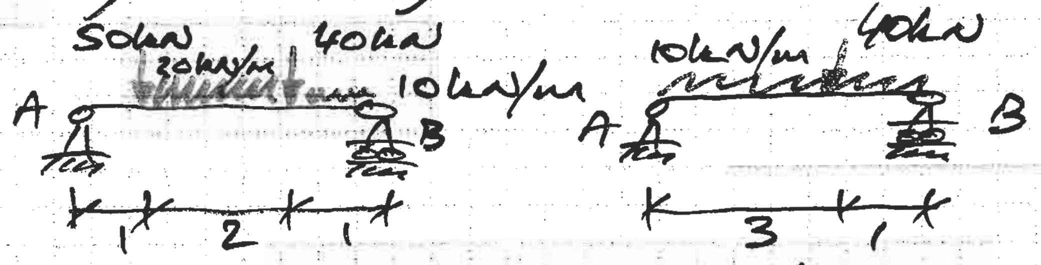
(4) 

Draw the I.L.'s of: V_A, V_B, V_D & M_E, M_B & M_F

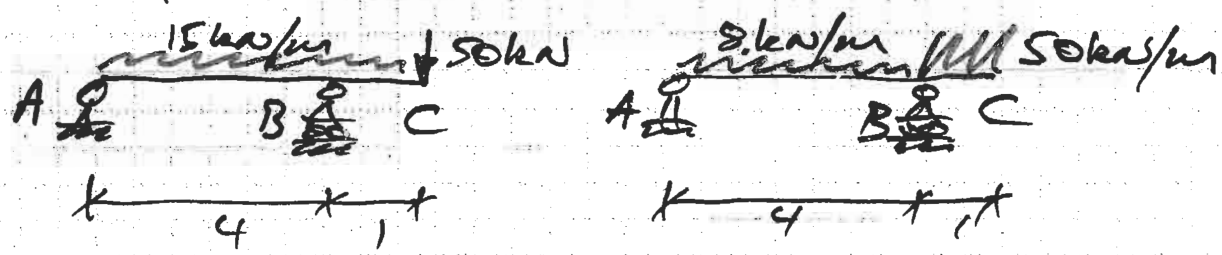
For a Partial UDL of 20 kN/m which can be any length, determine the maximum value of each of the load effects given.

For the following problems, use the influence lines calculated to determine the values of the load effect of interest for the following loading:

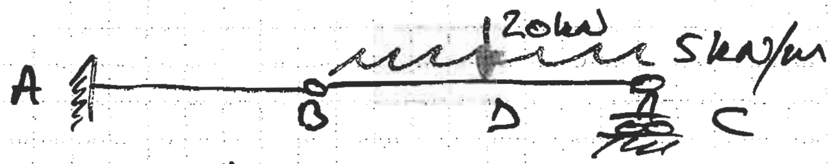
(5 & 6)



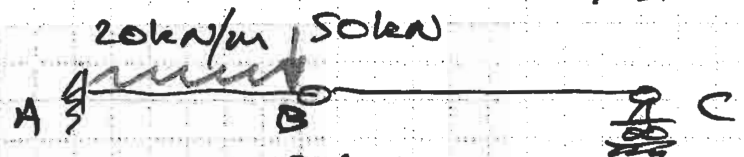
(7 & 8)



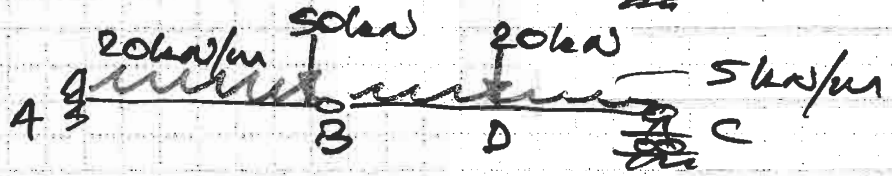
(9)



(10)



(11)



(12) The bridge of problem (4) has a characteristic dead load (G_k) of 40kN/m. This needs to be load-patterned by the MIN-MAX partial load factors of $0.9 G_k$ & $1.4 G_k$. Determine the worst-case patterns for each of the load effects and calculate their values.

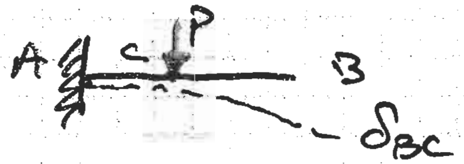
STRUCTURAL THEOREMS USED WITH I.L.'s

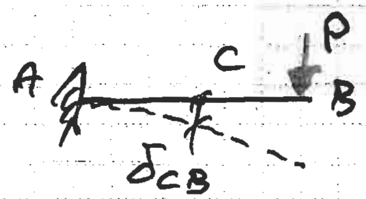
Maxwell's Reciprocal Theorem:

This states that the deflection in a structure at point X due to a load applied at point Y, is equal to the deflection at Y when the load is applied at X.

(This is a specific case of Betti's Theorem)

Example:



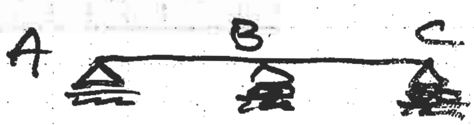


$$\boxed{\delta_{BC} = \delta_{CB}}$$

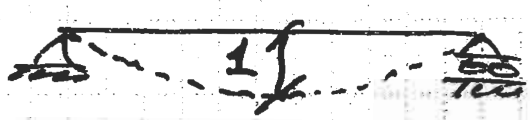
The Muller-Breslau Principle

This states that the ordinates of the I.L for any load effect are equal to those of the deflection curve obtained by releasing the restraint corresponding to the load effect and inducing a unit displacement, of the released load effect, at the same location, in the remaining structure.

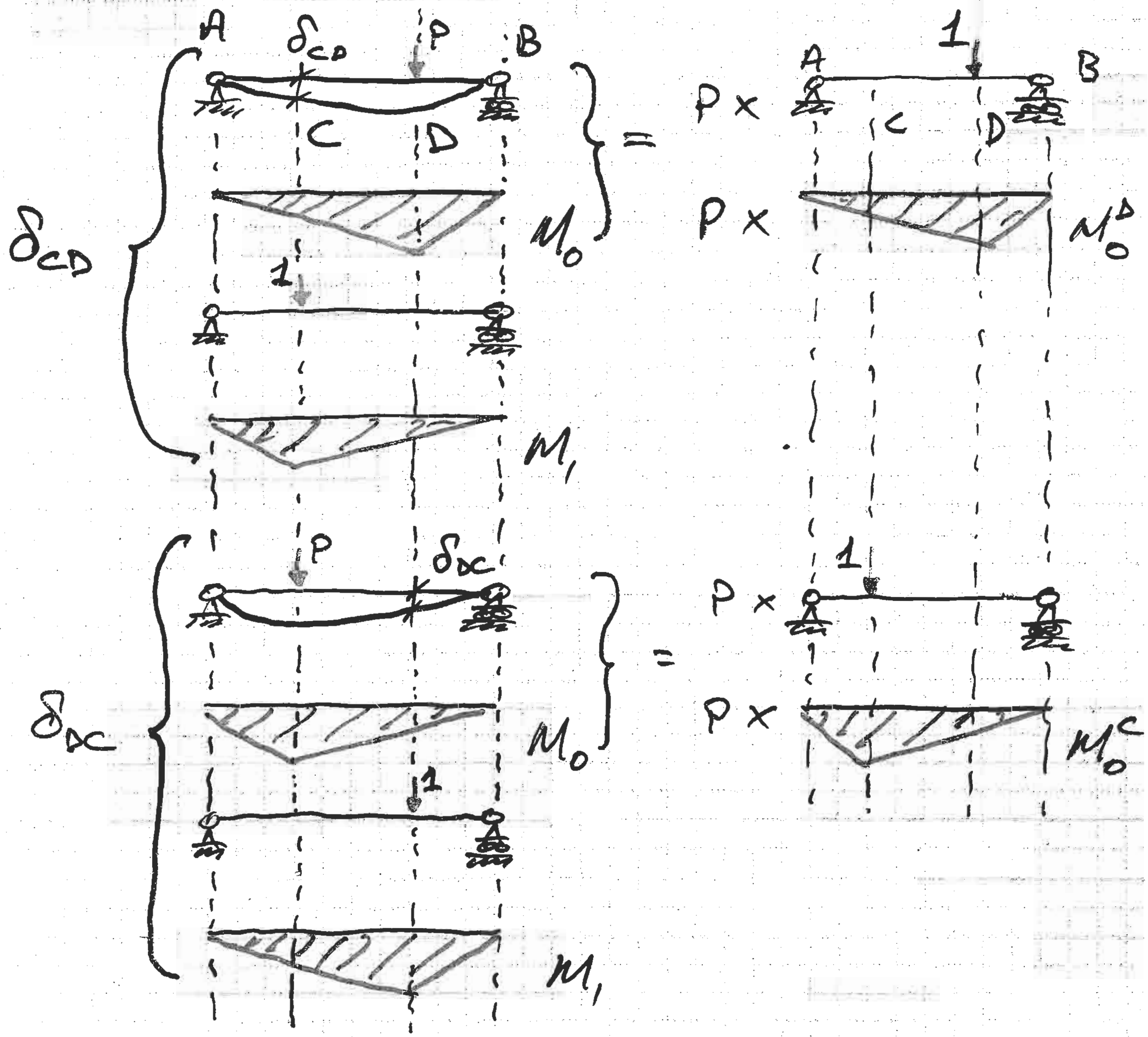
Example:



I.L. for V_B:



MAXWELL'S THEOREM BY VIRTUAL WORK



Let:

M_0^D = BMD of real (M_0) moments for a unit load at D

M_0^C = similar but 1 at C

M_1^D = BMD of virtual (M_1) moments for a unit (virtual) force at D

M_1^C = similar but for C

Recalling Virtual Work, we have:

EXT. V.W. = INT. V.W.

EXT. REAL DISPS × EXT. VIRTUAL FORCE = INT. REAL DISPS × INT. VIRTUAL FORCES

$$1 \times \delta = \int_0^L \Theta \times M_1$$

But, $\Theta = \frac{M_0}{EI} \cdot dx$

$$\Rightarrow \quad \boxed{\delta = \int_0^L \frac{M_1 M_0 \, dx}{EI}}$$

Using the Principle of Superposition:

$$M_0 \text{ for } P \text{ at } D = P \times M_0^D$$

$$M_0 \text{ for } P \text{ at } C = P \times M_0^C$$

Also, note that from the notation:

$$M_1 \text{ for } \delta_{CD} = M_1^C$$

$$M_1 \text{ for } \delta_{DC} = M_1^D$$

Thus we have,

$$\delta_{CD} = \int \frac{(M_0 \text{ for } P \text{ at } D)(M_1 \text{ for } 1 \text{ at } C)\,dx}{EI}$$

$$= \int \frac{(P M_0^D)(M_1^C)\,dx}{EI}$$

$$\Rightarrow \boxed{\delta_{CD} = \frac{P}{EI}\int M_0^D \cdot M_1^C \cdot dx}$$

Similarly we have:

$$\delta_{DC} = \int \frac{(P M_0^C)(M_1^D)\,dx}{EI}$$

$$\Rightarrow \boxed{\delta_{DC} = \frac{P}{EI}\int M_0^C \cdot M_1^D \cdot dx}$$

Note that even though one is a "real" BMD and the other a virtual BMD, the expressions in x for the moments (and the bending moment diagrams) are exactly the same, hence:

$$M_0^D = M_1^D = M^D \text{ say } \; \& \; M_0^C = M_1^C = M^C \text{ say}$$

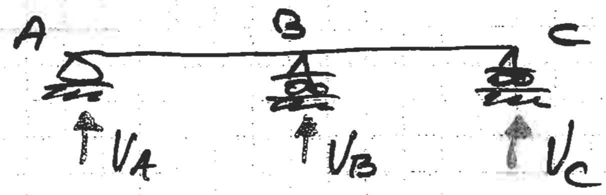
$$\Rightarrow \delta_{CD} = \frac{P}{EI}\int M^D \cdot M^C\,dx \; ; \; \delta_{DC} = \frac{P}{EI}\int M^C \cdot M^D \cdot dx$$

Hence,

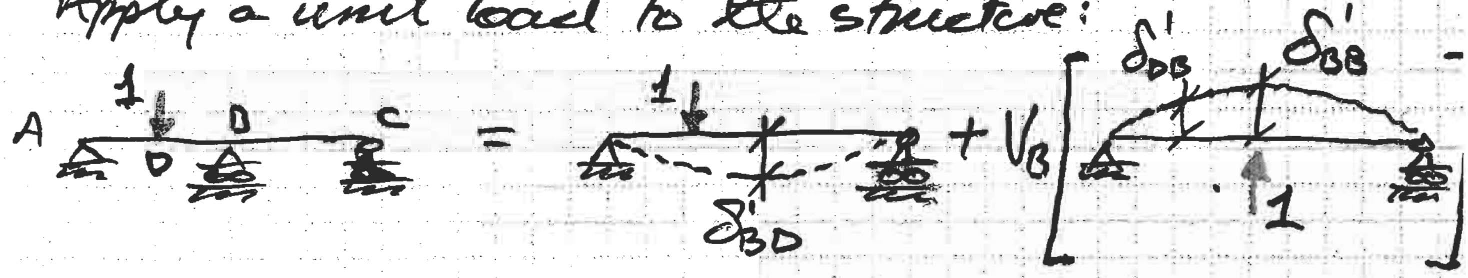
$$\boxed{\delta_{CD} = \delta_{DC}}$$

Influence Lines via Maxwell's Theorem

Calculate the I.L. for V_B in this beam:



Apply a unit load to the structure:



where the primes indicate that the deflection is due to a unit load and the second subscript gives the location of that load.

For compatibility of displacement of B:

$$\delta'_{BD} = V_B \cdot \delta'_{BB}$$

$$\Rightarrow \quad V_B = \delta'_{BD} / \delta'_{BB}$$

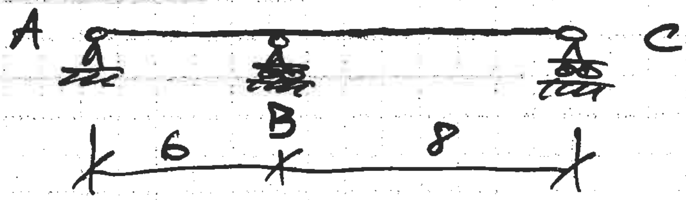
But by Maxwell's theorem:

$$\boxed{V_B = \delta'_{DB} / \delta'_{BB}}$$

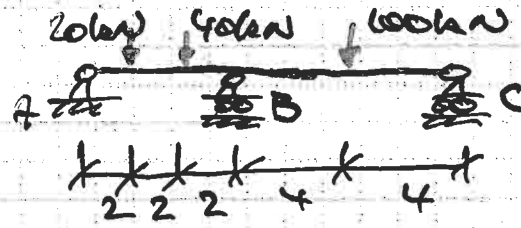
That is, the deflections along the structure normalized by the deflection at B. Further, the last expression shows that the load applied does not have to be a unit load because of the normalization that occurs.

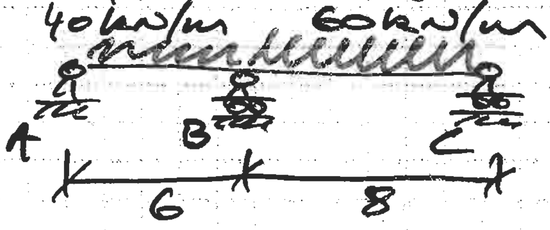
Example:

Draw the influence line for the vertical reaction at B; use 2m intervals.



Using the I.L., calculate the value of the reaction at B due to the following:

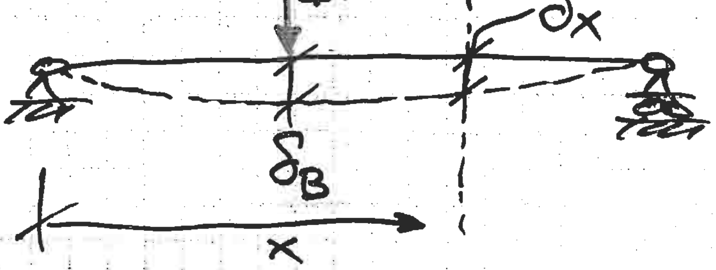
(1)


(2)


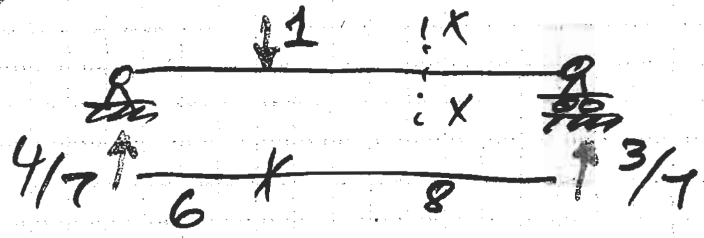
We see from Maxwell's theorem that:

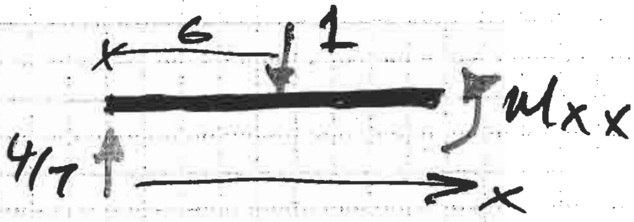
$$\text{I.L. for } V_B = \frac{\delta \text{ at } x}{\delta \text{ at } B} \text{ due to Unit load located at } B$$

So, replace V_B by a unit load, and calculate the deflection at 2m intervals along the beam:



Using Macaulays Method:





$$M_{xx} = EI\frac{d^2y}{dx^2} = \frac{4}{7}x - 1[x-6] \quad \text{—— ①}$$

$$\Rightarrow EI\frac{dy}{dx} = \frac{4x^2}{14} - \frac{1}{2}[x-6]^2 + C_1 \quad \text{—— ②}$$

$$EIy = \frac{4x^3}{42} - \frac{1}{6}[x-6]^3 + C_1x + C_2 \quad \text{—— ③}$$

Note that:

@ $x=0$, $y=0$, support A

@ $x=14$, $y=0$, support C

From ③, for $x=0, y=0$, $C_2=0$, for $x=14$, $y=0$:

$$\Rightarrow 0 = \frac{4(14)^3}{42} - \frac{8^3}{6} + 14C_1 \Rightarrow C_1 = \frac{-88}{7}$$

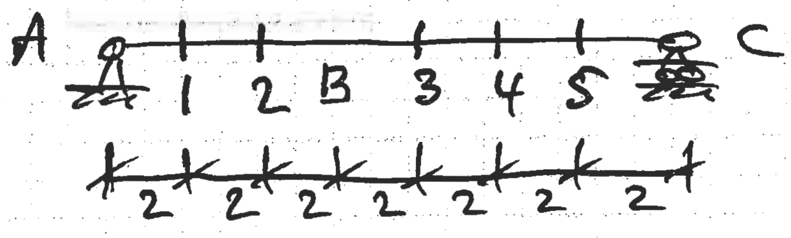
Thus,

(δ'_{xB}): $$EIy = \frac{4x^3}{42} - \frac{1}{6}[x-6]^3 - \frac{88}{7}x \quad \text{—— ④}$$

At B, $x=6$

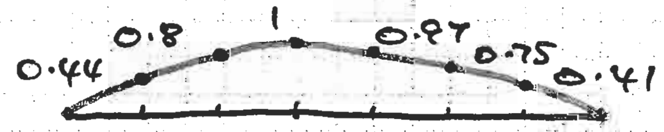
(δ'_{BB}) $$\Rightarrow EI\delta_B = \frac{4(6)^3}{42} - \frac{88}{7}(6) = \frac{-384}{7}$$

Using 2m intervals, we have:

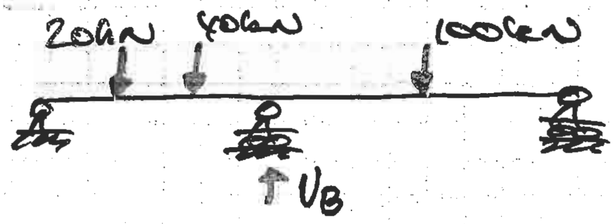


Location of Unit load	x (m)	$EI\delta'_{xB}$ from (4)	$V_B = \delta'_{xB}/\delta'_{BB}$
A	0	0	0
1	2	$-512/21$	0.44
2	4	$-928/21$	0.80
B	6	$-384/7$	1
3	8	$-1116/21$	0.97
4	10	$-864/21$	0.75
5	12	$-468/21$	0.41
C	14	0	0

Thus the I.L. for V_B is:



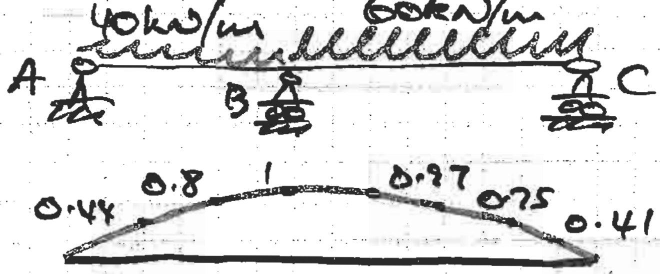
(1)



$V_B = 20(0.44) + 40(0.8) + 100(0.75) = \underline{115.8\text{kN}}$

With this information the structure can be solved for all other load effects.

(2)



I.L. for V_B

There are three ways to calculate the area under the I.L.:

(1) Approx – treat as trapezoids

(2) Closer – use Simpsons Rule

(3) Exact – integrate

Trapezoids:

Straight lines

The area of each trapezoid is : 

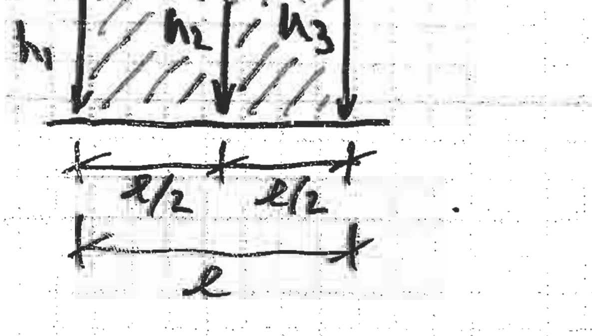
$= \left(\frac{h_1 + h_2}{2}\right) l$

$$\therefore V_B = 40\left[\left(\frac{0.44}{2}\right) + \left(\frac{0.44+0.8}{2}\right) + \left(\frac{0.8+1}{2}\right)\right] \times 2$$

$$+ 60\left[\left(\frac{1+0.97}{2}\right) + \left(\frac{0.97+0.75}{2}\right) + \left(\frac{0.75+0.41}{2}\right) + \left(\frac{0.41}{2}\right)\right] \times 2$$

$V_B = 454.8\,kN$ (1.6% out from exact value)

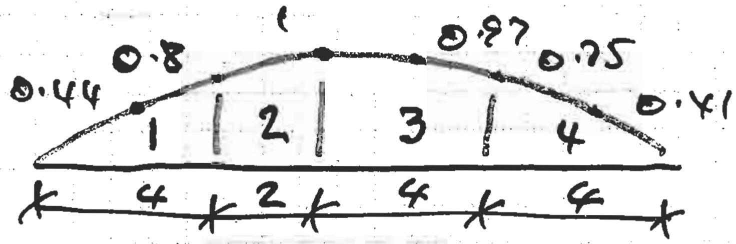
Simpsons Rule:



Shaded Area

$$= \frac{l}{6}(h_1 + 4h_2 + h_3)$$

Thus our I.L. can be broken up as follows:



$$V_B = 40\left[\tfrac{4}{6}(0 + 4(0.44) + 0.8)\right] \quad \text{— Area 1}$$

linear estimation

$$+\,40\left[\tfrac{2}{6}(0.8 + 4(0.9) + 1.0)\right] \quad \text{— Area 2}$$

$$+\,60\left[\tfrac{4}{6}(1 + 4(0.97) + 0.75)\right] \quad \text{— Area 3}$$

$$+\,60\left[\tfrac{4}{6}(0.75 + 4(0.41) + 0)\right] \quad \text{— Area 4}$$

$$= 40[1.707] + 40[1.8] + 60[3.753] + 60[1.593]$$

$V_B = 461\,kN$ (0.3% error from exact)

Integration

The expression for the I.L. is that of ④ divided by δ'_{BB}:

$$h(x) = \frac{-28}{16128}x^3 + \frac{7}{2304}[x-6]^3 + \frac{88}{384}x \quad \left[= \underset{(I.L.)}{V_B} = \frac{\delta'_{xB}}{\delta'_{BB}}\right]$$

Thus,

$$V_B = 40\int_0^6 h(x)\,dx + 60\int_6^{14} h(x)\,dx$$

As

$$\int h(x) = \frac{-7}{16128}x^4 + \frac{7}{9216}[x-6]^4 + \frac{88}{768}x^2$$

$$\Rightarrow V_B = 40[3.5625 - 0] + 60[8.895 - 3.5625]$$

$$\Rightarrow V_B = 462.45\,kN$$

The Muller-Breslau Principle

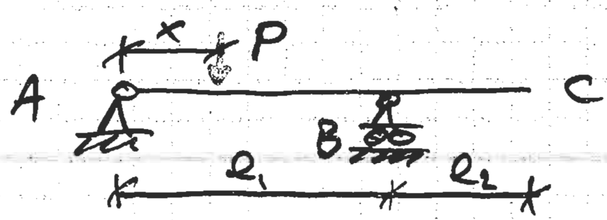
We can observe from the Reciprocal Theorem that if the deflection at B say (as in the example) was given a value of 1 initially, then the influence line for the reaction at B is immediately got:

$$\text{I.L. for } V_B = \frac{\delta'_{xB}}{\delta'_{BB}}$$

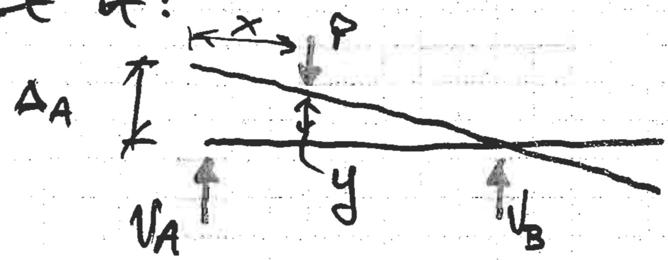
$\Rightarrow$ make $\delta'_{BB} = 1$

$\Rightarrow$ I.L. for $V_B = \delta(x)$

The same result can be arrived at through Virtual Work:



To find the reaction at A, we remove it and impose a small virtual displacement at it:



We note there is no internal virtual work as there is no internal virtual forces or displacements.

Thus,

$$\text{EXT. V.W.} = \text{INT. V.W.}$$

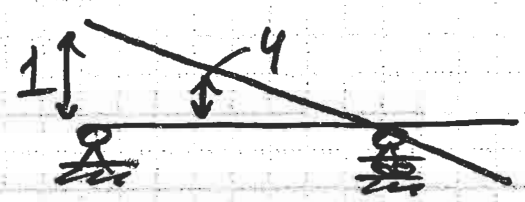
$$\Rightarrow \quad V_A \cdot \Delta_A - Py + V_B \cdot 0 = 0$$

$$\Rightarrow \quad V_A = P \cdot y / \Delta_A$$

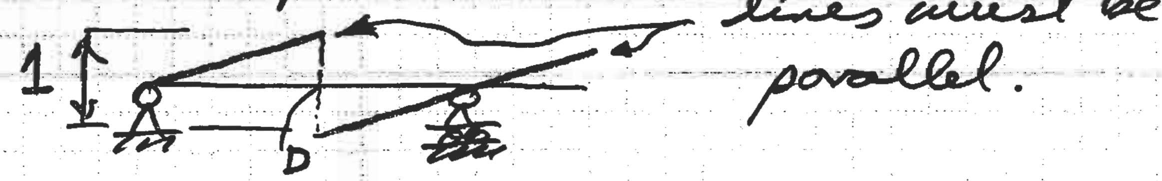
Exactly the result of the reciprocal theorem. Now if $P = 1$ (as in influence line analysis) and Δ_A is set to 1, then:

I.L. for $V_A = y$

when:

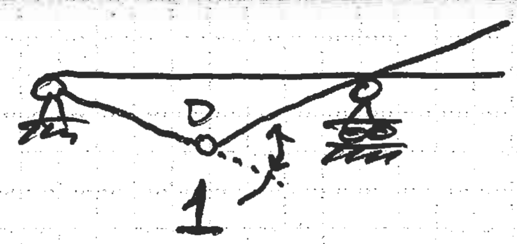


Similarly, for shear force we use a 1 unit "crank" displacement:



This is the I.L. for shear at D

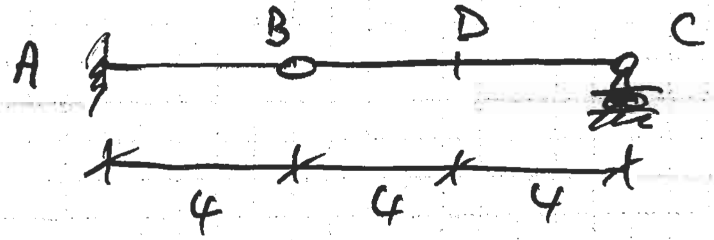
And for B.M., we introduce a unit rotation:



This is the I.L. for M at D.

Thus, an imposed unit displacement at the location of and in the sense of the released effect gives a deflected shape which is the appropriate I.L.

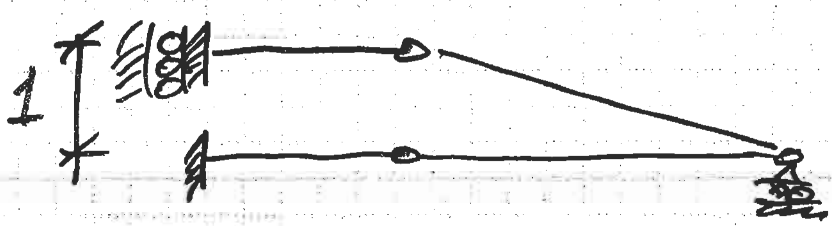
Example

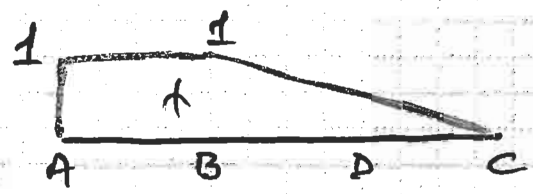


Using Muller-Breslau, find the I.L. for:

- V_A · V_C · M_A · M_D · V_B · V_D

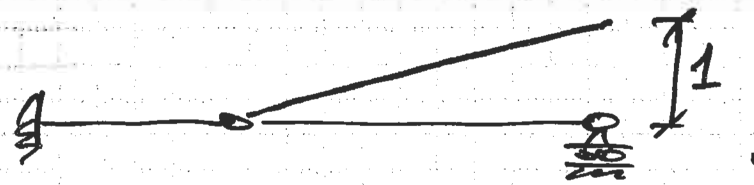
· V_A: Remove vertical restraint at A and impose a unit displacement:

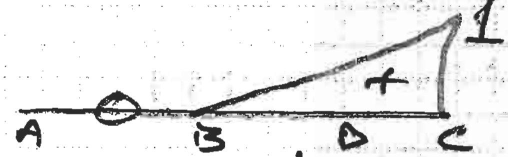




I.L. for V_A

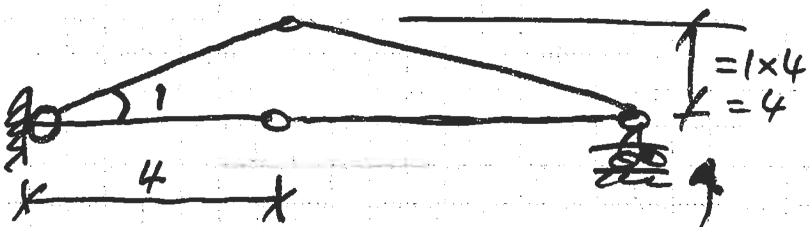
· V_C: Remove restraint V_C and give it 1:

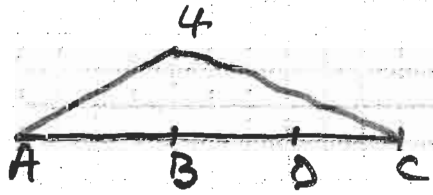




I.L. for V_C

· M_A: Remove rotational restraint and impose unit rotation:

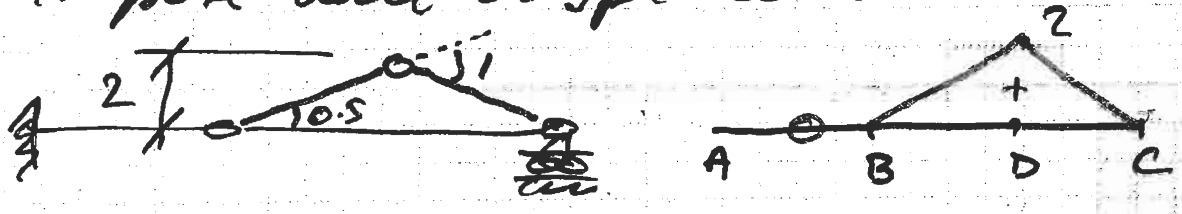




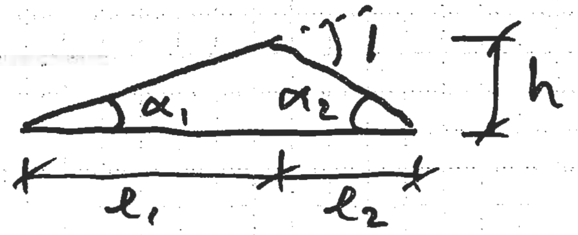
I.L. for M_A

$S = R\theta$ for small angles

• M_D Remove rotational restraint and impose unit displacement:



A difficulty lies in determining the height of the deflected shape. Note:



By small angles we know:

$$\alpha_1 l_1 = h \quad \& \quad \alpha_2 l_2 = h$$

Also, by opposite angles are equal:

$$\alpha_1 + \alpha_2 = 1$$

Thus it can be shown that

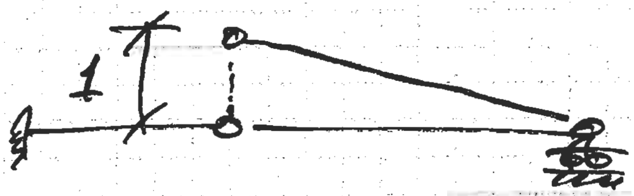
$$h = \frac{1}{(1/l_1 + 1/l_2)} \quad \text{or,} \quad \alpha_2 = \frac{l_1}{l_1 + l_2} \; ; \; \alpha_1 = \frac{l_2}{l_1 + l_2}$$

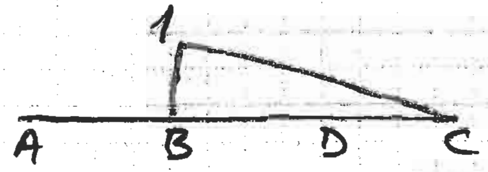
For symmetrical cases, $\alpha_1 = \alpha_2$, $l_1 = l_2$

$$\Rightarrow h = 0.5 l_1$$

This is as our case above.

• V_B As usual, impose unit displacement:

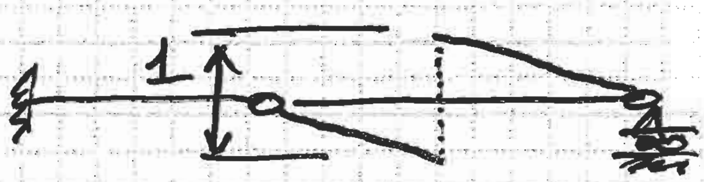


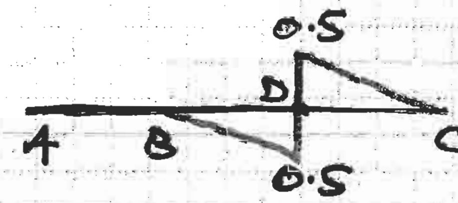


I.L. for V_B

Note that the tip of the cantilever AB will not deflect downwards as there is nothing to resist the force it would take to push it downwards.

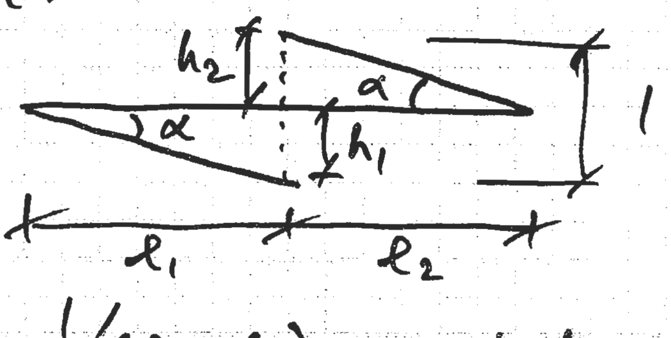
• V_D Again, impose a unit "crank" displacement:





I.L. for V_D

Note that as the two "bars" released must be parallel, their angles must be equal:



Thus, $\alpha = 1/(l_1+l_2)$ and $h_1 = \dfrac{l_1}{l_1+l_2}$; $h_2 = \dfrac{l_2}{l_1+l_2}$

These are not to be confused with those for the BM influence lines.

I.L.'s for Indet / det structures:

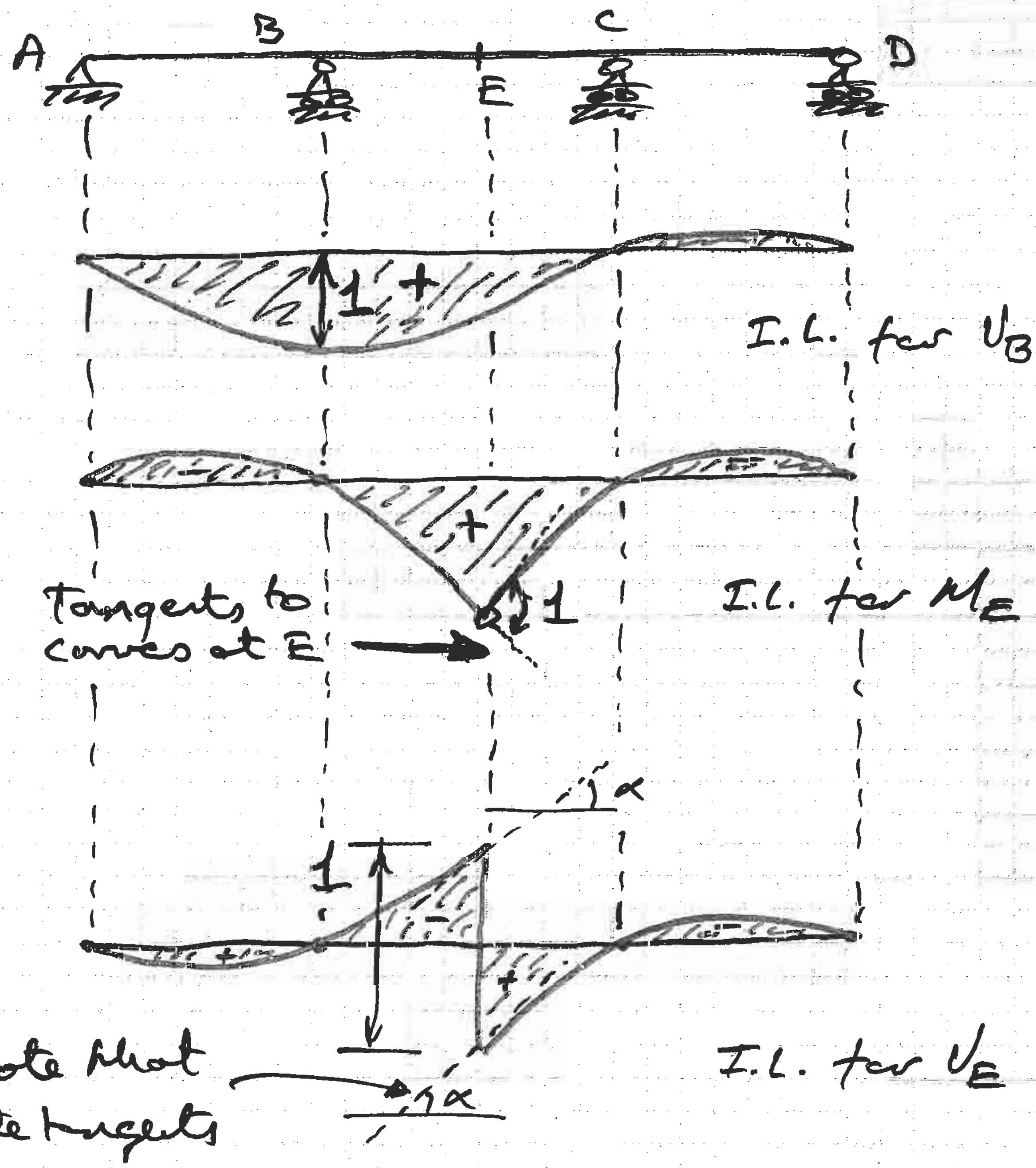
Given the Muller-Breslau Principle, we can see that removing a restraint has the effect of turning a determinate structure into a mechanism in which all elements of the structure then move as rigid elements, i.e. do not deform. Thus we can see that for statically determinate structures the I.L.'s will be composed of straight lines only, while those of statically indeterminate structures will be parabolic.

Note that as I.L.'s are often used to establish critical loading patterns, the actual values of the ordinates are not required, merely the shape.

If required though, the ordinates can be obtained by solution (as previously) or by computer analysis of the cut-back structure with the deflections normalized similar to the Reciprocal Theorem method.

Note that for statically indeterminate structures the same methods apply:

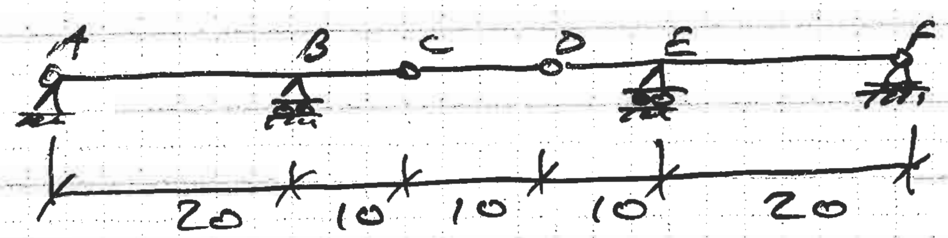
<u>Example</u>



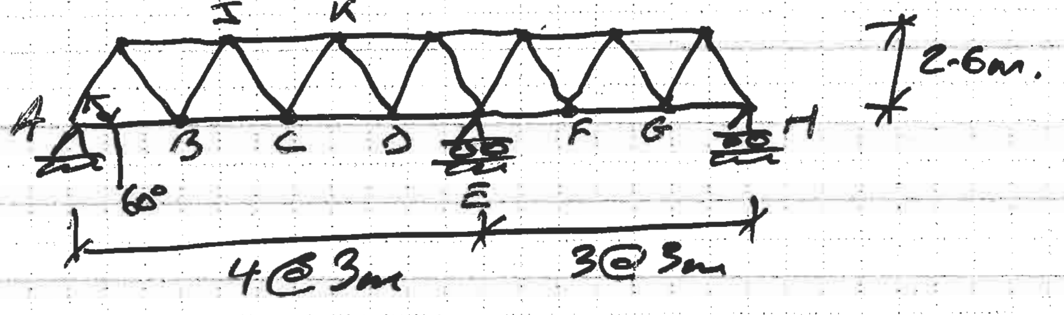
Note that the tangents to the curves are parallel.

Q5 '99S

(a)



(b)



(a) Draw influence lines for

(1) Reaction @ A
(2) Reaction @ B
(3) Shear @ C

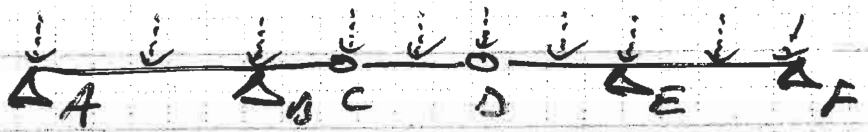
(b) A model of the truss supported at A & H gives deflections of:

Node	A	B	C	D	E	F	G	H
δ	0	10	16	21	27	20	13	0

(1) Plot I.L. for reaction @ E
(2) Det. reaction @ E for 120 kN @ B & 160 kN @ C
(3) Find Axial forces in JC & JK.

(a) — Ordinary method.

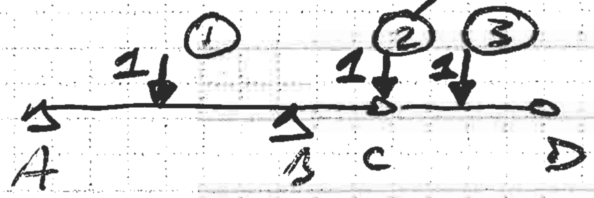
R_A, R_B, V_C



We could march the unit load across beam as shown. However, we can omit several of these steps:

- the beam is symmetrical
  $\Rightarrow$ Only work @ A to D
- loads over supports have no affect other than a unit reaction.

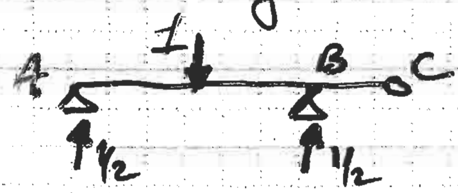
Thus we only have to analyse for:



Analyse 1

Note CD is a pin-pin member & no load transfer from AC to DF is possible for loads not on CD.

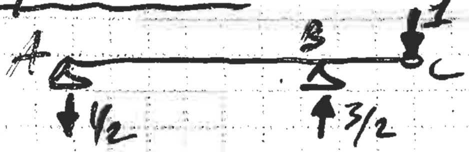
$\Rightarrow$



$R_A = 1/2$ $R_B = 1/2$

$V_C = 0$

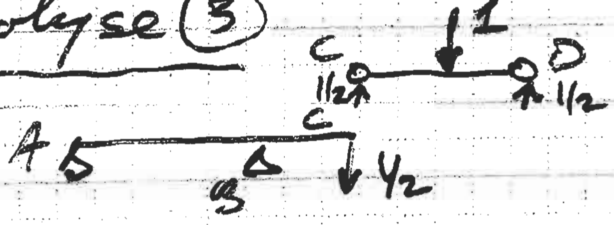
Analyse 2



$R_A = -1/2$ $R_B = 3/2$

$V_C = 1$

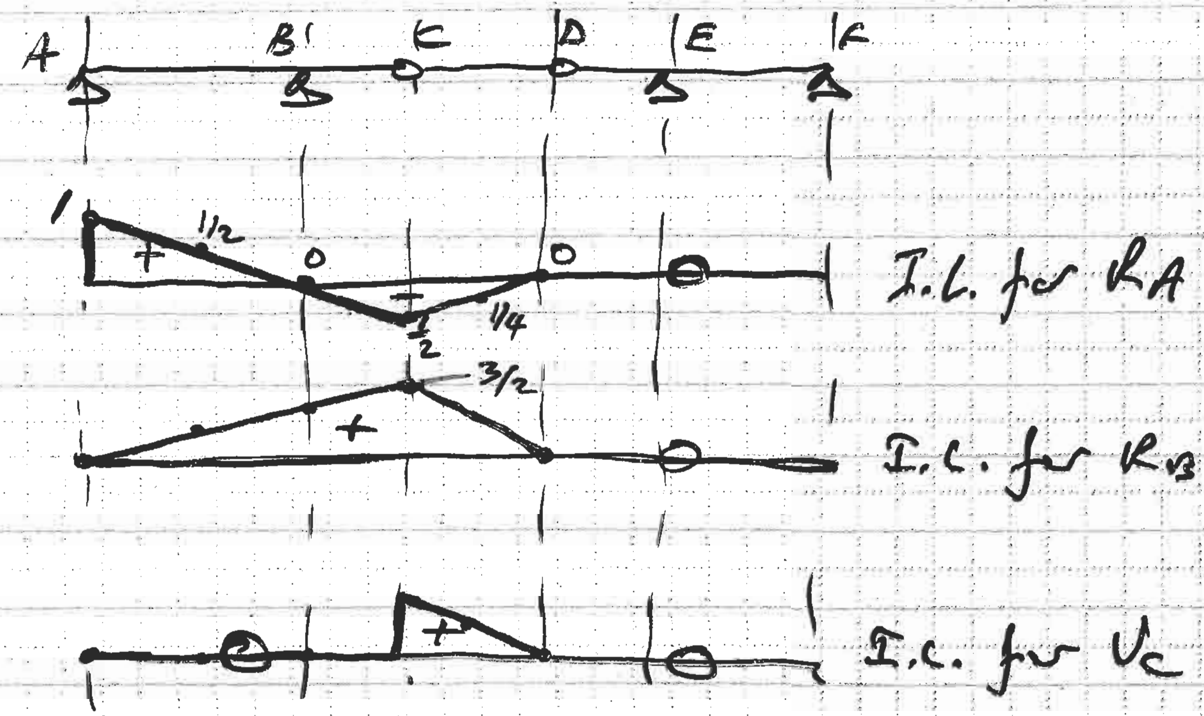
Analyse (3)



We see that the effect on AC is as per (2) but times 1/2

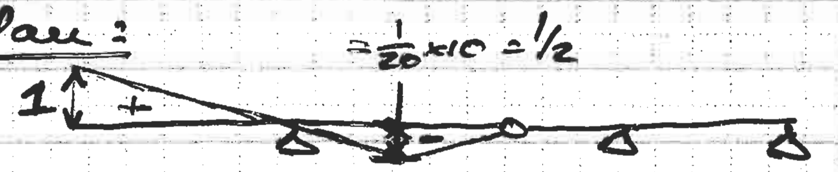
$\Rightarrow R_A = \frac{1}{2} \times (-\frac{1}{2}) = -\frac{1}{4} \quad R_B = \frac{3}{4} \quad V_C = \frac{1}{2}$

Plot the calculated values to get I.L.'s:

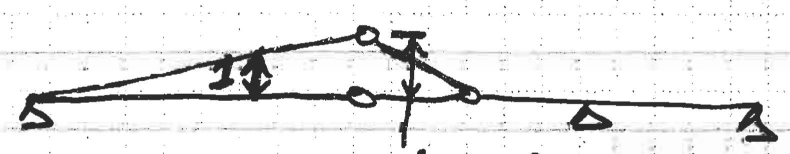


Muller-Breslau:

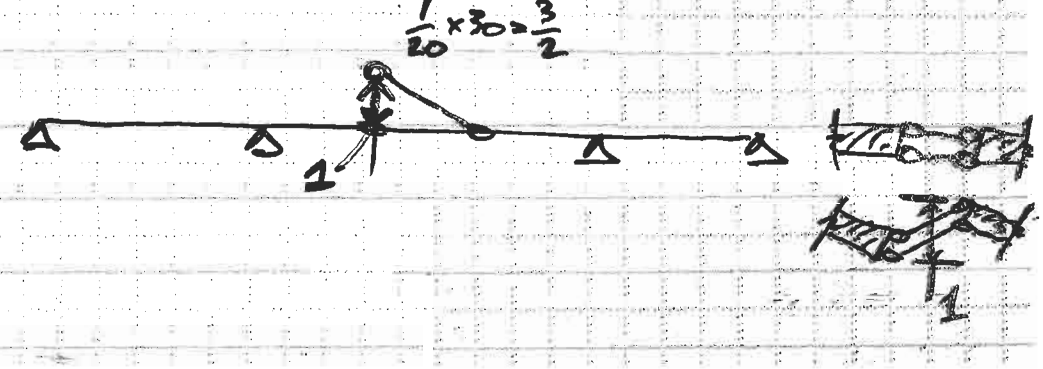
I.L. for R_A:



I.L. for R_B:



I.L. for V_C:



(b) (1)

Plot I.L. for R_E

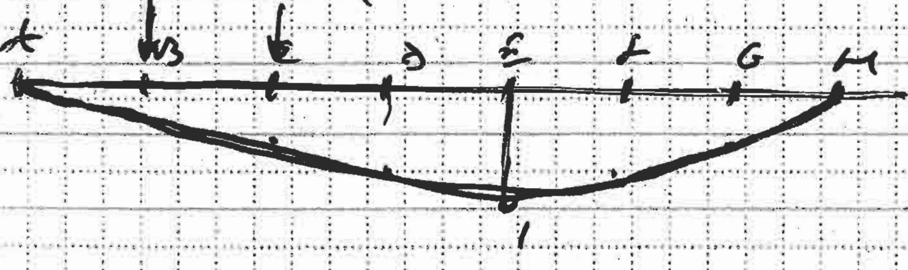
- Normalise deflection curve as per Muller-Breslau

⇒ I.L. value $R_E = \frac{\Delta_P}{\Delta_E}$

$\Delta_E = 27$ - given

⇒

Position	Δ	$R = \delta/\Delta_E$
A	0	0
B	10	0.37
C	16	0.59
D	21	0.78
E	27	1
F	20	0.74
G	13	0.48
H	0	0



I.L. for R_E.

(2)

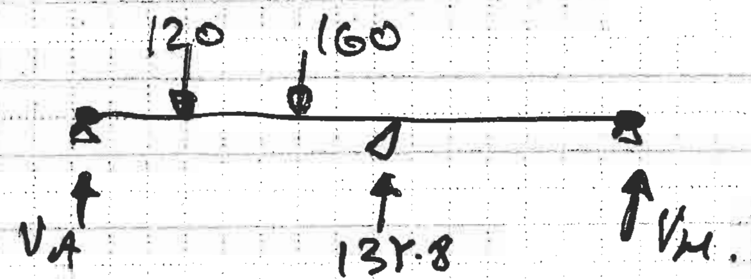
120kN @ B ⇒ 120×0.37

160kN @ C ⇒ 160×0.59

⇒ $R_E = \Sigma = 138.8$ kN

(3)

Thus:



ΣM about $M = 0$

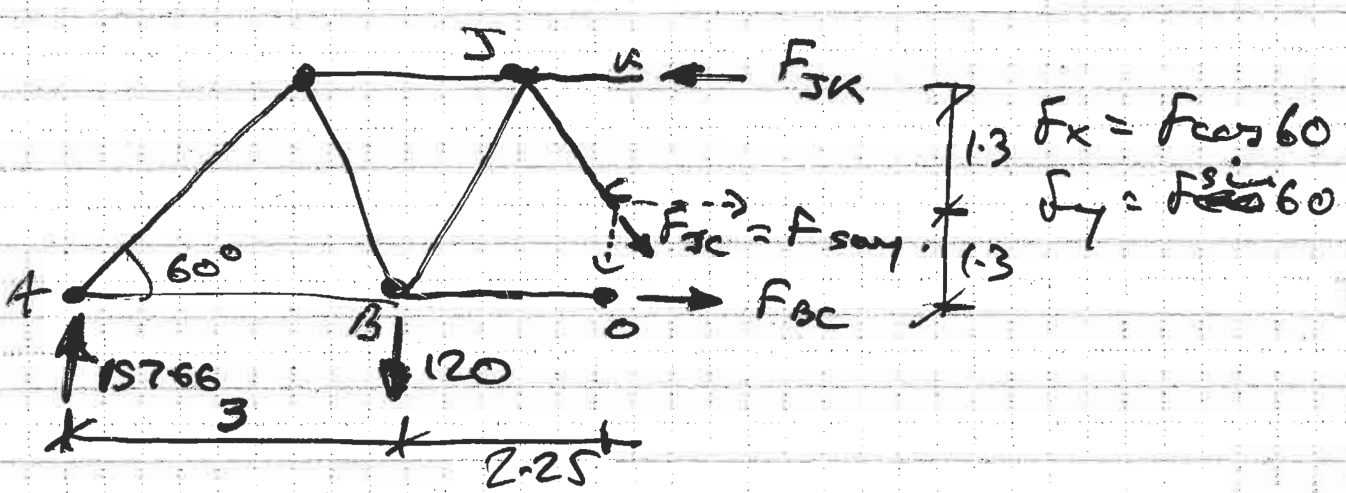
$\Rightarrow 138.8 \times 9 + V_A \times 21 - 160 \times 15 - 120 \times 18 = 0$

$\Rightarrow V_A = 157.66\,kN$

$\Sigma F_y = 0$

$\Rightarrow V_M = 120 + 160 - 157.66 - 138.8\,kN$

$= -16.46\,kN$ i.e. downwards.



ΣM about $C = 0$

$\Rightarrow 157.66 \times 5.25 - 120 \times 2.25 + 1.3 F_x - 2.6 F_{JK} = 0$ (1)

$\Sigma F_x = 0 \Rightarrow F_{JK} - F_x - F_{BC} = 0$ (2)

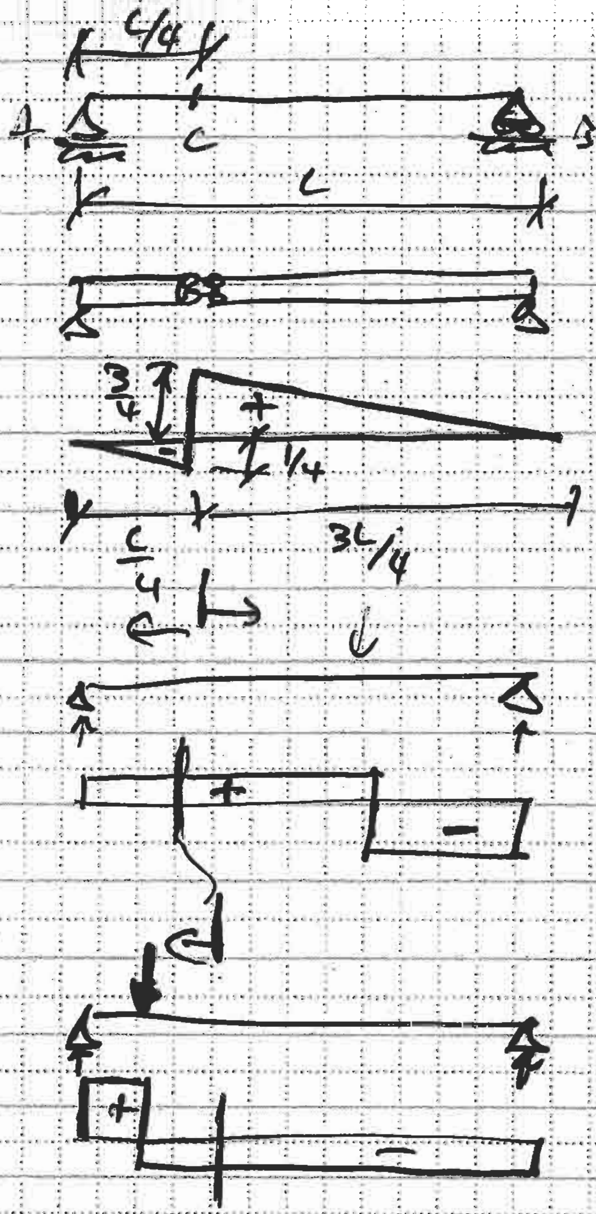
$\Sigma F_y = 0 \Rightarrow 157.66 - 120 - F_y = 0 \Rightarrow F_y = 37.66\,kN$.

$\Rightarrow \boxed{F_{JC} = F_y / \sin 60 = 43.48\,kN}$ (+)

$\Rightarrow F_x = F\cos 60 = 21.74\,kN$

$\Rightarrow$ from (1) $\Rightarrow \boxed{F_{JK} = 225.36\,kN}$

$\Rightarrow$ from (2) $\Rightarrow F_{BC} = 203.64\,kN$.



slopes are equal.

Muller-Breslau — Shear. I.C.'s.